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LETTER TO THE EDITOR

Car-oriented mean-field theory for traffic flow models

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Abstract. We present a new analytical description of the cellular automaton model for single-lane traffic. In contrast to previous approaches we do not use the occupation number of sites as dynamical variable but rather the distance between consecutive cars. Therefore certain longer-ranged correlations are taken into account and even a mean-field approach yields non-trivial results. In fact for the model with $v_{\max} = 1$ the exact solution is reproduced. For $v_{\max} = 2$ the fundamental diagram shows a good agreement with results from simulations.

Despite a large number of publications about the cellular automaton approach to traffic flow (see e.g. [1] and references therein) only a few of those deal with a systematic analytical description. Most works make use of large-scale computer simulations which can be carried out very efficiently for this class of models. Nevertheless, analytical results—exact or approximate—may give important information relevant for a complete understanding of those models.

The most important exact result is certainly the solution of the model for $v_{\max} = 1$ [2]. This result has been obtained using n-cluster approximation [2, 3], i.e. an improved mean-field theory taking into account correlations between n neighbouring sites. For $v_{\max} = 1$ the 2-cluster approximation is exact [2]. For higher velocities the n-cluster approximation for small n already yields very good results for the so-called fundamental diagram (flow-density relationship) [2, 3].

In computer simulation studies there are in principle two different approaches [3] called site-oriented and car-oriented¶. In the site-oriented approach the state of the system is specified by storing the state of each cell which can either be empty or occupied by a single car with velocity $v = 0, 1, \ldots, v_{\max}$. In the car-oriented approach, on the other hand, one stores the velocity of each car and the distance to the next car ahead.

Since the cluster approximation corresponds to a site-oriented approach, this analogy inspired us to investigate an analytical description based on the car-oriented approach, the so-called car-oriented mean-field theory (COMF)$^+$. The COMF already takes into account some longer-ranged correlations so that one can hope that it yields at least a good approximation.

For completeness we briefly repeat the definition of the CA model for single-lane traffic flow [5] in the following. The street is divided into L cells of a certain length (for realistic

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¶ In [3] this approach has been called particle-oriented.
$^+$ A brief account of some preliminary results has already been given in [4].

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applications 7.5 m) which can be occupied by at most one car or be empty. The cars have an internal parameter ('velocity') which can take on only integer values $v = 0, 1, 2, \ldots, v_{\max}$. The dynamics of the model are described by the following update rules for the velocities and the motion of cars [5]. In the first step all cars with velocities $v_i < v_{\max}$ are accelerated by one velocity unit, $v_i \to v_i' = v_i + 1$. The following step describes the slowing down due to other cars and prevents accidents. All cars with velocities $v_i' > d_i$ (where d_i is the number of free cells in front of car i) decelerate to velocity $v_i'' = d_i$. The last step in the velocity update is a randomization effect taking into accout several aspects of the driver's behaviour, e.g. fluctuations in driving style, overreaction at breaking, and retarded acceleration: every car with velocity $v_i'' > 0$ will slow down one unit with probability p, i.e. $v_i'' \xrightarrow{p} v_i''' = v_i'' - 1$. In the final step the car then moves v_i''' sites. These four rules, referred to as step 1 to step 4 in the following, are applied to all cars at the same time (parallel dynamics).

COMF for $v_{max} = 1$

We denote the probability to find at time t (exactly) n empty sites in front of a vehicle by $P_n(t)$. As in [2, 3] we change the order of the update steps to 2–3–4–1. This change has to be taken into account when calculating the flux $f(c, p)$. It has the advantage that after step 1 there are no cars with velocity 0, i.e. all cars have velocity 1. The time evolution of the probabilities $P_n(t)$ can conveniently be expressed through the probability $g(t)$ ($\bar{g}(t) = 1 - g(t)$) that a car moves (does not move) in the next timestep.

In order to find the time evolution of the $P_n(t)$ we first determine from which configurations at time t a given state at time $t + 1$ could have been evolved under the rules 2–3–4–1. Take for instance a car—called second car in the following—which has $n > 1$ free sites in front, i.e. its distance to the next car ahead (called first car in the following) is $n + 1$ sites. This configuration might have evolved from four different configurations at time t, depending on whether (i) both cars moved in the timestep $t \to t+1$ (which happens with probability $qg(t)$), (ii) both cars did not move (with probability $p\bar{g}(t)$), (iii) only the first car moved (with probability $pg(t)$), or (iv) only the second car moved (with probability $q\bar{g}(t)$). This means that the second car at time t had either n free site in front (cases (i) and (ii)), or $n - 1$ free sites (case (iii)), or $n + 1$ free sites (case (iv)).

The special cases $n = 0, 1$ can be treated in a analogous fashion. In this way one obtains the time evolution of the probabilities as

$$P_0(t+1) = \bar{g}(t)[P_0(t) + qP_1(t)] \tag{1}$$

$$P_1(t+1) = g(t)P_0(t) + [qg(t) + p\bar{g}(t)]P_1(t) + q\bar{g}(t)P_2(t) \tag{2}$$

$$P_n(t+1) = pg(t)P_{n-1}(t) + [qg(t) + p\bar{g}(t)]P_n(t) + q\bar{g}(t)P_{n+1}(t). \tag{3}$$

A car will move in the next timestep if there is at least one empty cell in front of it (probability $\sum_{n\geqslant 1} P_n(t)$) and if it does not decelerate in the randomization step 3 (probability $q = 1 - p$). Therefore, the probabilities $g(t)$ and $\bar{g}(t)$ are given by

$$g(t) = q\sum_{n\geqslant 1} P_n(t) = q[1 - P_0(t)]$$

$$\bar{g}(t) = P_0(t) + p\sum_{n\geqslant 1} P_n(t) = p + qP_0(t) \tag{4}$$

where we have used the normalization

$$\sum_{n\geqslant 0} P_n(t) = 1. \tag{5}$$

The probabilities can also be related to the density $c = N/L$ of cars. Since each car which has the distance n to the next one in front of it 'occupies' $n+1$ sites we have the following relation:

$$\sum_{n\geqslant 0}(n+1)P_n(t) = \frac{1}{c}. \tag{6}$$

Here we are mainly interested in the stationary state ($t \to \infty$) with $\lim_{t\to\infty} P_n(t) = P_n$. In order to determine the probabilities in the stationary state we introduce the generating function

$$P(z) = \sum_{n=0}^{\infty} P_n z^{n+1}. \tag{7}$$

After multiplying the corresponding equation in (1)–(3) by z^{n+1} and summing over all equations one finds an explicit expression for the generating function,

$$P(z) = \frac{q(\bar{g}+gz)zP_0}{q\bar{g}-pgz}. \tag{8}$$

The normalization condition (5) and the density relation (6) imply that the generating function has to satisfy

$$P(1) = 1 \qquad \text{and} \qquad P'(1) = \frac{1}{c} \tag{9}$$

where $P'(z)$ denotes the derivative of $P(z)$.

Using (8) it is easy to obtain the probabilities explicitly:

$$\begin{aligned} P_0 &= \frac{2qc-1+\sqrt{1-4qc(1-c)}}{2qc} \\ P_n &= \frac{P_0}{p}\left(\frac{pg}{q\bar{g}}\right)^n = \frac{P_0}{p}\left(\frac{p(1-P_0)}{P_0+p(1-P_0)}\right)^n \qquad (n\geqslant 1) \end{aligned} \tag{10}$$

where we have already used (6) to express P_0 through the density c of cars.

To obtain the fundamental diagram we have to calculate the flux. It is given by $f(c,p) = cg = qc(1-P_0)$ from which one recovers the exact result [2, 3]

$$f(c,p) = \frac{1-\sqrt{1-4qc(1-c)}}{2}. \tag{11}$$

In [2, 3] we expressed the exact solution in terms of the pair probabilities $P(n_j, n_{j+1})$ to find two neighbouring sites j and $j+1$ in the state (n_j, n_{j+1}). Here $n_j = 0$ denotes an empty site and $n_j = 1$ a site occupied by a car (with velocity 1)†. In [2, 3] it was shown that probabilities for larger clusters factorize, i.e. $P(n_1, \ldots, n_L) = \prod_{j=1}^{L-1} P(n_j, n_{j+1})$. The 2-cluster probabilities are related to the P_n through

$$\begin{aligned} P(1,1) &= cP_0 \\ P^2(1,0) &= c(1-c)P_1 \\ P(0,0) &= (1-c)\frac{P_{n+1}}{P_n} \qquad (n\geqslant 1). \end{aligned} \tag{12}$$

The factors c and $1-c$ appear due to the different normalization of the P_n and $P(n_j, n_{j+1})$. The P_n are normalized by the number of cars whereas the $P(n_j, n_{j+1})$ are normalized by the number of sites.

† Cars with velocity 0 do not exist after the acceleration step.

The fact that the COMF yields the exact result is not unexpected since the COMF takes into account all relevent correlations for the case $v_{\max} = 1$ where only nearest-neighbour correlations are non-trivial [2, 3]. Although the model for $v_{\max} > 1$ is know to exhibit a different behaviour [3] the COMF might yield good results in this case because it takes into longer-ranged correlations. In the following we will investigate the case $v_{\max} = 2$.

COMF for $v_{max} = 2$

The case $v_{\max} = 2$ can be treated in a similar way as $v_{\max} = 1$. However, it is now necessary to introduce two different functions $P_n(t)$ and $B_n(t)$ describing the probabilities to find exactly n empty sites in front of a car with velocity 1 and 2, respectively. Proceeding as in the case $v_{\max} = 1$ we find the evolution equations. For the stationary state the probabilities obey the equations

$$P_0 = g_0[P_0 + B_0] \tag{13}$$

$$P_1 = g_1[P_0 + B_0] + pg_0[P_1 + B_1] \tag{14}$$

$$P_2 = g_2[P_0 + B_0] + pg_1[P_1 + B_1] + pg_0P_2 \tag{15}$$

$$P_3 = pg_2[P_1 + B_1] + pg_1P_2 + pg_0P_3 \tag{16}$$

$$P_n = pg_2P_{n-2} + pg_1P_{n-1} + pg_0P_n \qquad (n \geqslant 4) \tag{17}$$

and

$$B_0 = qg_0[P_1 + B_1 + B_2] \tag{18}$$

$$B_1 = qg_1[P_1 + B_1 + B_2] + g_0[qP_2 + pB_2 + qB_3] \tag{19}$$

$$B_2 = qg_2[P_1 + B_1 + B_2] + g_1[qP_2 + pB_2 + qB_3] + g_0[qP_3 + pB_3 + qB_4] \tag{20}$$

$$B_n = g_2[qP_{n-1} + pB_{n-1} + qB_n] + g_1[qP_n + pB_n + qB_{n+1}] + g_0[qP_{n+1} + pB_{n+1} + qB_{n+2}] \qquad (n \geqslant 3). \tag{21}$$

The probabilities g_α that a car moves α sites ($\alpha = 0, 1, 2$) in the next timestep are given by

$$\begin{aligned} g_0 &= P_0 + B_0 + p\sum_{n\geqslant 1} P_n + pB_1 \\ g_1 &= q\sum_{n\geqslant 1} P_n + qB_1 + p\sum_{n\geqslant 2} B_n \\ g_2 &= q\sum_{n\geqslant 2} B_n. \end{aligned} \tag{22}$$

We just mention here that it is possible to derive the identities $g_0 = \sum_{n\geqslant 0} P_n$ and $g_1 + g_2 = \sum_{n\geqslant 0} B_n$ from (22) and (13)–(21). Using the normalization

$$\sum_{n\geqslant 0}[P_n + B_n] = 1 \tag{23}$$

we have $g_0 + g_1 + g_2 = 1$.

The conservation of density leads to the constraint

$$\sum_{n\geqslant 0}(n+1)[P_n + B_n] = \frac{1}{c}. \tag{24}$$

We introduce the generating functions

$$P(z) = \sum_{n=0}^{\infty} P_n z^{n+1} \tag{25}$$

$$B(z) = \sum_{n=0}^{\infty} B_n z^{n+1}. \tag{26}$$

These functions have to satisfy $P(1) + B(1) = 1$ and $P'(1) + B'(1) = 1/c$ due to (23) and (24), respectively.

After multiplication with z^{n+1} and summation over all the equations (13)–(21) one finds

$$P(z) = \frac{g(z)}{1 - pg(z)}[(qP_0 + B_0)z + pB_1z^2] \tag{27}$$

$$B(z) = \frac{zg(z)}{z^2 - (q + pz)g(z)}[qP(z) - qB_0 - (qP_0 + pB_0 + qB_1)z + (q - p)B_1z^2] \tag{28}$$

where we have introduced the function $g(z) = g_0 + g_1z + g_2z^2$. Note that this function satisfies $g(1) = 1$ and $g'(1) = (1/c)f(c, p)$ is just the average velocity of the vehicles.

If one expresses the sums appearing in (22) by $P(z = 1)$ and $B(z = 1)$ one obtains the following relations:

$$B_1 = \frac{q}{p}g_0 - \frac{1}{p}(B_0 + qP_0) \tag{29}$$

$$g_1 = p(1 - P_0) + \left(q - \frac{p}{q}\right)B_0 + \left(1 - \frac{p}{q}\right)B_1 \tag{30}$$

$$g_2 = q(1 - P_0) - (1 + q)B_0 - B_1. \tag{31}$$

With these relations the normalization condition $P(1) + B(1) = 1$ is also satisfied. Now we can express the generating function completely in terms of the two probabilities P_0 and B_0 only, since $g_0 = P_0/(P_0 + B_0)$ from (13).

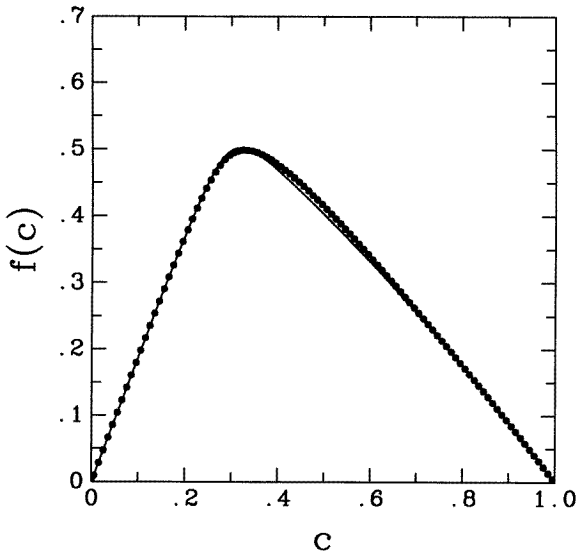

Figure 1. Fundamental diagram for $v_{max} = 2$ and $p = 0.1$. The comparison of the COMF result (full curve) with results from computer simulations (●) shows an excellent agreement.

At this point it is surprising that we are still left with two unknowns P_0 and B_0 since we only have one free parameter, the density c. In the following we determine a relation between P_0 and B_0 from analytic properties of $B(z)$. Thus the generating functions

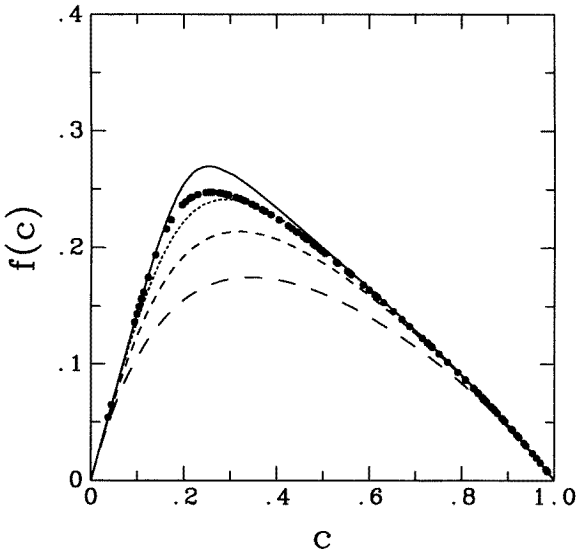

Figure 2. Fundamental diagram for $v_{\max} = 2$ and $p = \frac{1}{2}$. The full curve is the COMF result. For comparison the results from computer simulations (●) and the n-cluster approximation for $n = 1, 2, 3$ (broken curves) are also shown.

depend only on one free parameter, e.g. P_0, which then can be related to the density via $P'(1) + B'(1) = 1/c$.

The denominator of $B(z)$ can be rewritten as $z^2 - (q + pz)g(z) = pg_2(1-z)(z-s_+)(z-s_-)$, where $s_\pm$ are given by

$$s_\pm = \frac{1}{a}\left[1 \pm \sqrt{1 + \frac{qg_0}{pg_2}a^2}\right]$$

with $a = 2pg_2/(g_0 + qg_1)$. Thus the denominator of $B(z)$ has three zeros, $z = 1$ and $z = s_\pm$. Two of these are located in the unit circle since $|s_-| \leqslant 1$. These zeros have to be cancelled by corresponding zeros of the numerator since $B(z)$ has to be analytic in the unit circle (otherwise one would not have $\lim_{n\to\infty} B_n = 0$). It is easy to see that the numerator indeed has a zero at $z = 1$. Demanding that it also has a zero at $z = s_-$ we find the missing relation between P_0 and B_0,

$$qP(s_-) - qB_0 - (qP_0 + pB_0 + qB_1)s_- + (q - p)B_1 s_-^2 = 0. \tag{32}$$

Owing to (24) we can regard the generating functions $P(z)$ and $B(z)$ as functions of the density c only. The fundamental diagram can then be obtained using the following expression for the flux:

$$f(c, p) = c[g_1 + 2g_2]. \tag{33}$$

Results are shown in figure 1 for $p = 0.1$ and figure 2 for $p = \frac{1}{2}$. For $p = 0.1$ we find an excellent agreement of the two curves. For $p = 0.5$ we still find an excellent agreement for small densities ($c < 0.2$) and high densities ($c > 0.5$). Only near the maximum are there deviations. For comparision we have also included in figure 2 results from the site-oriented approach, i.e. the n-cluster approximation. The COMF result is much better than the 2-cluster result and comparable to the 3-cluster approximation. It seems that the COMF tends to overestimate the flux whereas the n-cluster approximation yields a lower bound for the

flux [3]. The result for $p = 0.1$ shows, however, that the COMF result is not systematically larger than the simulation results.

For small densities the average distance between the cars is large. Therefore correlations between cars and neighbouring empty sites are much more important than those between two cars. These correlations are better described by the COMF which is the reason why the COMF is superior to the cluster approach for small cluster sizes in this regime.

The agreement between simulations and the COMF result is very good up to densities close to the critical density $c_{\mathrm{crit}} \approx 0.19$ for $p = \frac{1}{2}$ [6]. Here correlations between distances become important. It is to be expected that a combination of the cluster approach and the COMF will give a much better agreement of the calculated fundamental diagram and the simulations in the region near the critical point and the flux maximum.

We have also applied the COMF to generalizations of some modified models [7–9]. The models discussed in [7, 8] have different modified acceleration rules which for $v_{\max} = 1$ break the 'particle–hole' symmetry, i.e. the fundamental diagrams are no longer symmetric with respect to $c = 0.5$. It turns out that the COMF gives good agreement with simulations, but is no longer exact (even for $v_{\max} = 1$). A full account of these results will be given elsewhere.

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